Light-polarization tunneling in optically active media

2008 J. Phys. A: Math. Theor. 41 035301

(http://iopscience.iop.org/1751-8121/41/3/035301)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 035301 (10pp) doi:10.1088/1751-8113/41/3/035301

Light-polarization tunneling in optically active media

Robert Botet[1] **and Hiroshi Kuratsuji**[2]

[1] Laboratoire de Physique des Solides Bât.510, CNRS UMR8502/Université Paris-Sud,
Centre d'Orsay, F-91405 Orsay, France
[2] Department of Physics, Ritsumeikan University-BKC, Kusatsu City 525-8577, Japan

E-mail: botet@lps.u-psud.fr and kra@kitaoji.com

Received 29 August 2007, in final form 29 November 2007
Published 4 January 2008
Online at stacks.iop.org/JPhysA/41/035301

Abstract

Evolution of the polarization state of light transmitted through an optically active medium follows from the Maxwell theory of electromagnetism. The theory can be reduced to the study of a Schrödinger-like equation for two levels representing the right- and left-circular polarizations, respectively. Using quantum mechanical techniques, we show that the Stokes parameters should exhibit tunneling in the anisotropic nonlinear medium—a phenomenon similar to *quantum* tunneling—provided the nonlinear parameter be large enough. In order to recover the quantum results in the classical framework, one has to consider additional fluctuations of the initial parameters, since the classical problem has no fundamental fluctuations. Indeed, even for small fluctuations, the polarization state may exhibit *chaotic* tunneling across classically forbidden regions. This is exemplified through polarization tunneling in a nonlinear transparent medium submitted to constant Kerr effect and modulated nonlinear parameter.

PACS numbers: 42.25.Ja, 42.65.Sf, 05.45.−a

1. Introduction

Polarization is one of the main characteristics of a transverse wave traveling through our usual 3D space [1]. Particularly, the quantity is dramatically sensitive to the history of the interactions that the wave experienced during travel [2].

An alternative and fruitful way to study the polarization of a plane electromagnetic wave is to map the Maxwell equations onto a 2-states Schrödinger equation [3]. The space-evolution of the polarization is then analogous to the problem of time-evolution of a 2-states quantum system. For the latter, a number of results are yet known, as the problem corresponds to important and active topics in the recent physics, such as Josephson effect for neutral atoms [4], quantum dynamics of atomic Bose–Einstein condensate in a double-well potential [5], dynamics of mesoscopic quantum spins [7], 4-momentum evolution of a negatively-charged massless relativistic particle in electromagnetic field [6], etc. As a result of the formal equivalence, any result in one field can be translated into results in another context. This was recently exemplified with the theoretical evidence for the existence of Rabi oscillations of the circular polarization of light traveling through a dielectric medium embedded in a constant magnetic field plus a helicoidal electric field [8, 9]. The problem is the optical analogue to the nuclear magnetic resonance [10].

In this work, we investigate a novel phenomenon of light polarization tunneling, which is realized as the tunneling of the Stokes vector through polarization states which are classically forbidden. This is closely connected with chaotic behavior of the Stokes parameters when the optical medium parameters are submitted to small fluctuations.

2. General quantum mechanical framework

Consider a monochromatic electromagnetic plane wave arriving onto a thick transparent slab under normal incidence. We consider pure polarization states, such that the wave polarization can be defined unambiguously as the combination of right- and left-hand circular polarization components. The normal incidence defines the z-axis, while the x- and y-directions complete the orthogonal frame. The slab is generally a nonlinear anisotropic dielectric medium, with magnetic permeability $\mu = 1$. The z-axis is considered as the principal axis of the dielectric tensor $\hat{\epsilon}$, such that $\hat{\epsilon}$ is essentially a 2×2 tensor written here as

$$\hat{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}, \tag{1}$$

and the transverse wave propagates inside the medium following the z-direction, with the wavelength λ in vacuum.

2.1. The Schrödinger-like equation for the circular-polarized modes

Applying the para-axial approximation [11] to the general second-order Maxwell–Helmholtz equation for the transverse electric field (E_x, E_y), one obtains a 2-states Schrödinger-like equation, which writes simply when the circular polarization basis is used. Namely [12],

$$\mathrm{i}\lambda \frac{\mathrm{d}\psi}{\mathrm{d}z} = \hat{\mathcal{H}}\psi, \tag{2}$$

with the 2-components complex vector $\psi = (\psi_+, \psi_-) \equiv ((E_x + \mathrm{i}E_y)/\sqrt{2}, (E_x - \mathrm{i}E_y)/\sqrt{2})$. The quantity $|\psi_+|^2$ (resp. $|\psi_-|^2$) represents then the intensity of the right- (resp. left-) hand circularly polarized component of the wave. The Hamiltonian $\hat{\mathcal{H}}$ in (2) must be Hermitian, since the norm of the vector ψ is conserved in a transparent medium, and traceless [3] because the two eigenvalues of the operator $\hat{\mathcal{H}}$ are opposite.

If the dielectric tensor (1) is the Hermitian matrix [13], the Hamiltonian $\hat{\mathcal{H}}$ is given by the 2×2 matrix:

$$\hat{\mathcal{H}} = \begin{pmatrix} c & a - \mathrm{i}b \\ a + \mathrm{i}b & -c \end{pmatrix}, \tag{3}$$

with the three real components [8]:

$$a = \pi \frac{\epsilon_{yy} - \epsilon_{xx}}{\epsilon_{xx} + \epsilon_{yy}}, \qquad b = \pi \frac{\epsilon_{xy} + \epsilon_{yx}}{\epsilon_{xx} + \epsilon_{yy}}, \qquad c = \mathrm{i}\pi \frac{\epsilon_{xy} - \epsilon_{yx}}{\epsilon_{xx} + \epsilon_{yy}}.$$

Generally, the coefficients a, b, c depend on the polarization state ψ for the nonlinear media.

Within this approach, we define the Stokes parameters $(s_i)_{i=0,\ldots,3}$ as the real numbers given by

$$s_0 = |\psi_+|^2 + |\psi_-|^2, \tag{4}$$

$$s_1 = \psi_-\psi_+^\star + \psi_+\psi_-^\star, \tag{5}$$

$$s_2 = \mathrm{i}\big(\psi_+\psi_-^\star - \psi_-\psi_+^\star\big), \tag{6}$$

$$s_3 = |\psi_+|^2 - |\psi_-|^2. \tag{7}$$

Since the medium is transparent, the value of s_0 is conservative.

2.2. *Functional integral approach for the Stokes parameters*

We reformulate the evolution of optical polarization as a quantum problem [14]. Let us introduce the normalized 2-components quantum state $|\psi\rangle \equiv |\psi_1, \psi_2\rangle$. The quantity $|\psi_1|^2$ (resp. $|\psi_2|^2$) represents the occupation probability for the system to be in the state 1 (resp. 2). We define then the Stokes parameters $(s_i)_{i=0,\ldots,3}$ as pseudo-spin variable with components

$$s_i = \langle\psi^\dagger|\sigma_i|\psi\rangle, \tag{8}$$

where the σ_i are the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indeed, the parameters defined by (8) are identical to the definitions (4)–(7). Then, we write the path integral expression [14] for the transition amplitude between states $|\psi\rangle$ and $|\psi'\rangle$, following the original idea of Feynman [15]:

$$\langle\psi'^\dagger|\psi\rangle = \int \mathrm{e}^{\mathrm{i}\mathcal{I}/\hbar}\mathcal{D}(\langle\psi'^\dagger|, |\psi\rangle), \tag{9}$$

where the action functional $\mathcal{I}$, associated with the path, is given by

$$\mathcal{I} = \int \langle\psi^\dagger|\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \hat{\mathcal{H}}\right)|\psi\rangle\,\mathrm{d}t, \tag{10}$$

with t the physical time and $\hat{\mathcal{H}} = a\sigma_1 + b\sigma_2 + c\sigma_3$ the Hamiltonian operator corresponding to (3). The Dirac action principle $\delta\mathcal{I} = 0$ gives the usual Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{\mathcal{H}}|\psi\rangle,$$

formally identical to (2), where the Planck constant $\hbar$ plays the role of the wavelength λ and the time t the role of the spatial coordinate z.

Expressing now the state $|\psi\rangle$ within the spinor representation:

$$\psi_1 = \sqrt{s_0}\cos(\theta/2), \qquad \psi_2 = \sqrt{s_0}\sin(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi},$$

the Stokes parameters write

$$s_1 = s_0\sin\theta\cos\phi, \qquad s_2 = s_0\sin\theta\sin\phi, \qquad s_3 = s_0\cos\theta, \tag{11}$$

and the Lagrangian of the quantum field $|\psi\rangle$ is

$$\begin{aligned} \mathcal{L} &= \mathrm{i}\hbar\langle\psi^\dagger|\frac{\partial}{\partial t}|\psi\rangle - \langle\psi^\dagger|\hat{\mathcal{H}}|\psi\rangle \\ &= -\hbar s_0\dot{\phi}\sin^2(\theta/2) - \mathcal{H}, \end{aligned} \tag{12}$$

where the dot means the t-derivative, and the operator $\mathcal{H} = (a\cos\phi + b\sin\phi)s_0\sin\theta + cs_0\cos\theta$ (note that the coefficients a, b, c may depend on θ and ϕ in the nonlinear case). Furthermore, the first term in $\mathcal{L}$, which is called the *canonical term*, and denoted by $\mathcal{L}^c$, is written in terms of the three-dimensional Stokes vector $\vec{s} \equiv (s_1, s_2, s_3)$ as

$$\mathcal{L}^c = \frac{\hbar}{2}\frac{s_2\dot{s}_1 - s_1\dot{s}_2}{s_0 + s_3}.$$

From the variational principle $\delta\mathcal{I} = 0$, one can derive equations of motion for the Stokes parameters, through the Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 0; \qquad \frac{\partial\mathcal{L}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0.$$

There are essentially two ways to express the differential equations governing the evolution of the Stokes parameters. If one writes the Hamiltonian of the system as $\mathcal{H} = H_1(\phi, s_3)$, one derives the Hamilton's canonical equations (after replacing θ by s_3 through (11))

$$\frac{\hbar}{2}\dot{\phi} = \frac{\partial H_1}{\partial s_3}, \tag{13}$$

$$\frac{\hbar}{2}\dot{s}_3 = -\frac{\partial H_1}{\partial\phi}, \tag{14}$$

which show that the two variables ϕ and s_3 are canonical for the problem [7].

The other way is to write the Hamiltonian of the system as $\mathcal{H} = H_2(s_1, s_2, s_3)$, yielding the evolution of $\vec{s}$,

$$\frac{\hbar}{2}\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{\mathcal{B}} \times \vec{s}, \tag{15}$$

with the pseudo-magnetic field $\vec{\mathcal{B}} = (\partial H_2/\partial s_1, \partial H_2/\partial s_2, \partial H_2/\partial s_3)$.

3. Tunneling of the Stokes parameters in the functional integral approach

Therefore, tunneling of the quantum state representing the circular polarization of the light in the optically active medium appears to be a standard problem within the framework of the functional integral approach. Let us consider the t-evolution of the state $|\psi\rangle$. According to the initial condition, the state of the system can be in different domains. Suppose that the phase-space is composed of two disconnected domains, say region 1 and region 2, separated by a (classically-)forbidden region. From (9), one can write generally the transition amplitude from the region 1 to the region 2 as

$$K_{1\to2} = \mathrm{e}^{\mathrm{i}/\hbar\int_1^2\mathcal{L}\mathrm{d}t},$$

in which the classical action has been replaced by the integral of the Lagrangian $\mathcal{L}$ of the system. The probability of tunneling between these two disconnected regions is given as

$$P_{1\to2}^{qu} = |K_{1\to2}|^2,$$

which can be smaller than 1 whenever the Lagrangian $\mathcal{L}$ takes imaginary values.

Translated into the initial electromagnetic problem, we have to change $\hbar$ into λ, and t into z, in order to recover the physical parameters. It results in the formula

$$P_{1\to2}^{qu} = \left|\mathrm{e}^{\mathrm{i}/\lambda\int_1^2\mathcal{L}\mathrm{d}z}\right|^2 \tag{16}$$

for the tunneling probability between the regions 1 and 2, where, according to formula (12),

$$\mathcal{L} = -s_0(\lambda\phi'\sin^2(\theta/2) + (a\cos\phi + b\sin\phi)\sin\theta + c\cos\theta),$$

and the $'$ denotes the z-derivative. Furthermore, expression (16) can be written in terms of the canonical term $\mathcal{L}_c$ as far as the orbit on the energy surface $\mathcal{H} = E$ is concerned, namely,

$$P^{qu}_{1\to 2} = \left|\mathrm{e}^{\mathrm{i}/\lambda \int_1^2 \mathcal{L}_c\,\mathrm{d}z}\right|^2.$$

To fix the ideas, let us take the values $(a, b, c) = (a, 0, gs_3)$, as a general set of values for (a, b, c) for optical medium with Kerr effect (a) and nonlinear effect $(g \neq 0)$ [8]. Dynamics of the Stokes components in the linear case $g = 0$ was treated in details in [2]. The medium is supposed to be homogeneous, that is the coefficients a and g are independent of the z-coordinate. We will discuss here the case where the initial state of the wave is partially circular-polarized, that is $(s_i)_{i=0,\ldots,3}|_{z=0} = (s_0, s_{10}, 0, s_{30})$, with $s_{10}^2 + s_{30}^2 = s_0^2$. Because of (4), intensity $s_0 > 0$. Then, $H_2 = as_1 + gs_3^2$, and the equations of motion (15) for the Stokes vector $\vec{s}$ write

$$\frac{\lambda}{2}\frac{\mathrm{d}\vec{s}}{\mathrm{d}z} = \begin{pmatrix} -2gs_2s_3 \\ -as_3 + 2gs_1s_3 \\ as_2 \end{pmatrix}, \tag{17}$$

with the conservative norm $||\vec{s}||^2 = s_0^2$.

The system being homogeneous, one can eliminate s_2 from the above equations, and express the component s_1 as a simple function of s_3:

$$s_1 - s_{10} = \frac{g}{a}\left(s_{30}^2 - s_3^2\right), \tag{18}$$

which reflects conservation of the Hamiltonian. Moreover, all the quantities of interest can be written as a function of s_3 only, for example,

$$\begin{aligned} -\frac{\lambda^2}{4}s_3'^2 &= F(s_3) \equiv g^2\left(s_{30}^2 - s_3^2\right)\left(\kappa^2 - s_3^2\right), \\ \mathcal{L}_{12}(s_3) &= g\left(s_{10}(a/g - s_{10})\frac{s_3}{s_0 + s_3} + s_3^2 - s_0s_3\right), \end{aligned} \tag{19}$$

with the coefficient κ^2 depending on the initial conditions, as $\kappa^2 = s_0^2 - (a/g - s_{10})^2 \leqslant s_0^2$, and where $\mathcal{L}_{12}$ is the value of the Lagrangian along the trajectory in the $\vec{s}$-space. The value of κ^2 may be either positive or negative according to the values of the ratio a/g and to the initial conditions.

Changing from (19) z for s_3 into the integral (16), one obtains the tunneling probability as

$$P^{qu}_{1\to 2} = \exp\left(\int_1^2 \frac{\mathcal{L}_{12}(s_3)}{\sqrt{F(s_3)}}\,\mathrm{d}s_3\right). \tag{20}$$

In formula (20), the integration range is over the values of s_3 for which the function F takes positive values. Indeed, there is no contribution of the negative values of $F(s_3)$ to $P^{qu}_{1\to 2}$, because of the modulus in definition (16). This 'classically' forbidden region depends on the initial state of the wave through the parameter s_{30} in equation (19). It is worth noting that (20) does not depend on the wavelength λ. So, we can expect the result to be valid independently of any approximation related to the proper value of λ.

3.1. Example

Let us consider the particular case where $a > 0$ (positive Kerr effect), and $g > 0$. The latter condition can be considered without loss of generality because the equations (17) are invariant by the replacement $(s_1, s_2, s_3) \longleftrightarrow (-s_1, s_2, s_3)$, except for the sign of g. The function $F(s_3)$, as defined in (19) may take positive values for particular values of the ratio

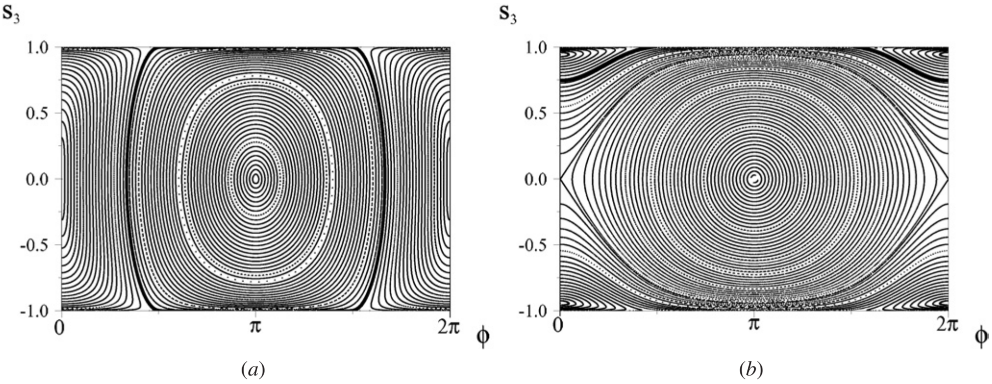

Figure 1. Poincaré sections for $s_0 = 1$ and (*a*) $(a, gs_0) = (1, 1/2)$; (*b*) $(a, gs_0) = (1, 3/2)$. Sixty-four regularly spaced initial conditions are used, and 3000 points are plotted for each trajectory. One sees in (*a*) the limit of stability of the fixed point $\{\phi = 0, s_3 = 0\}$. In (*b*), for which $gs_0 > a/2$, the localization of the system state around the two stable fixed points $\{\phi = 0, s_3 = \pm 2\sqrt{2}/3\}$ is exemplified through the trajectories (bold lines) of $\{\phi, s_3\}$ when the initial conditions $s_{10} = 0, s_{30} = +1$ are used (the corresponding equation is $a\cos\phi = g\sqrt{s_0^2 - s_3^2}$).

a/g, namely, $0 < a/g < s_0 + s_{10}$. Moreover, the tunneling probability (20) can be readily expressed as the sum of complete elliptic integrals. For example, whenever $\kappa^2 > 0$, one has

$$\frac{1}{2}\ln P^{qu}_{1\to 2} = -s_{30}\left[K\left(\kappa^2/s_{30}^2\right) - E\left(\kappa^2/s_{30}^2\right)\right] \\ + s_{10}(a/g - s_{10})\left[K\left(\kappa^2/s_{30}^2\right) - \Pi\left(\kappa^2/s_0^2|\kappa^2/s_{30}^2\right)\right], \tag{21}$$

with the usual notations for the complete elliptic integrals K, E, Π [17]. Note that the proper value of $P^{qu}_{1\to 2}$ depends explicitly on the initial conditions s_{10} and s_{30}, as well as on the intensity s_0.

For instance, for $s_0 = 1, s_{10} = 0$ and $g/a = 3/2$, the region $-\sqrt{5}/3 < s_3 < \sqrt{5}/3$ is classically forbidden as it would lead from (19) to negative values of $s_3'^2$ (see also figure 1). Nevertheless, formula (21) gives a positive transition probability $P^{qu}_{1\to 2} \simeq 0.07$ for tunneling from the region $s_3 \simeq 1$ to the region $s_3 \simeq -1$. The proper physical meaning for such an event in the classical case will be clarified in the following section.

4. General classical chaotic framework

As previously, we consider the case of an active medium with positive Kerr and nonlinear effects, that is the Hamiltonian $H_1(\phi, s_3) = a\sqrt{s_0^2 - s_3^2}\cos\phi + gs_3^2$.

From (13) and (14), the angles θ and ϕ verify the coupled differential equations:

$$\frac{\lambda}{2}\theta' = -a\sin\phi, \tag{22}$$

$$\frac{\lambda}{2}\phi' = -a\cot\theta\cos\phi + 2gs_0\cos\theta, \tag{23}$$

and $0 < \theta < \pi, 0 \leqslant \phi \leqslant 2\pi$. When $\sin\theta = 0$, equation (23) is replaced by $\lambda\phi'/2 = gs_0$.

Relations (22) and (23) can be regarded here as evolution equations for the couple of classical angular variables (θ, ϕ). In particular, all information about natural quantum fluctuations of the variables is wiped out. From this point of view, the discussion below—about the structure of the solutions of (22) and (23)—should be understood within the framework of classical physics, and possible fluctuations studied as an independent feature.

4.1. The fixed points

Four couples $\{\phi, s_3\}$ can be the fixed points of the equations above.

- The point $\{\phi = \pi, s_3 = 0\}$ is a stable fixed point for any value of the parameters a and g.
- The point $\{\phi = 0, s_3 = 0\}$ is another fixed point. Linearization of (22) and (23) around this point gives

  $$\ddot{\epsilon} + \frac{4a}{\lambda^2}(a - 2gs_0)\epsilon = 0.$$

  This fixed point is then stable when $gs_0 < a/2$ and unstable otherwise.
- By similar argument, one can show that the two points $\left\{\phi = 0, s_3 = \pm\sqrt{s_0^2 - a^2/4g^2}\right\}$ exist and are stable whenever $gs_0 > a/2$.

Bifurcation of the solution in $gs_0 = a/2$ is analogous to the critical behavior of the Lipkin spin model, namely, an ensemble of spins interacting identically one with each other [16]. Indeed, if one considers the action (10) with the Hamiltonian $H_2 = as_1 + gs_3^2$, one can re-interpret s_3 as the component of the sum of many spins over the third coordinate, while s_1 is the projection of this sum over the first coordinate (and a is then the magnetic field along the first coordinate). Within this framework, s_0 corresponds to the number of interacting spins. The partition function of such a model is

$$Z = \int \exp(-\beta\mathcal{H})\mathcal{D}\mathbf{S}. \tag{24}$$

When s_0 is large, the value of Z in (24) is given by the stationary phase condition, which is nothing but the equations of motion (23). The most probable configuration of the spin system is then $s_3 = 0$ when $gs_0 < a/2$, while it is $s_3 = \pm\sqrt{s_0^2 - a^2/4g^2}$ when $gs_0 \geqslant a/2$.

Note that the partition function above is obtained as the short wavelength limit $\lambda \to 0$ of the imaginary-time version of the functional integral used in (9), with the action functional $\mathcal{I} = -\int(\lambda s_0\phi' \sin^2(\theta/2) + \mathcal{H})\,\mathrm{d}z$. If we write $\mathcal{H} = s_0\tilde{\mathcal{H}}$ and consider large value for s_0, then the stationary phase condition leads to $\delta\tilde{\mathcal{H}} = 0$, which corresponds to a critical bifurcation between the two types of solutions.

4.2. Poincaré sections

All the fixed points can be visualized throughout the Poincaré sections for the canonical variables $\{\phi, s_3\}$ whose evolution is governed by the system of equations (22) and (23). We use the stroboscopic method with the frequency $2a/\lambda$, corresponding to the natural frequency of the linear (i.e. $g = 0$-) case.

The Runge–Kutta RK4 method is used with the increment of time small enough to lead to similar plots when the increment $\mathrm{d}z$ is multiplied or divided by a factor 5. Quantitative comparisons with known exact trajectories were also done (e.g., figure 1). Actually, $2a\,\mathrm{d}z/\lambda = 10^{-3}$ is chosen. In figures 1(*a*) and (*b*), one sees how the Poincaré sections change when the nonlinearity parameter g increases in magnitude.

4.3. Localization

The relation (18) leads directly to the localization of the vector $\vec{s}$, as can be written in terms of the couple of variables $\{\phi, s_3\}$:

$$a\sqrt{s_0^2 - s_3^2}\cos\phi + gs_3^2 = A, \tag{25}$$

with the constant $A = as_{10} + gs_{30}^2$. Let us consider a trajectory starting from $\{\phi = 0, s_3 = s_{30}\}$. This trajectory will cross the $s_3 = 0$ line iff $|A/a| \leqslant s_0$. This means that if $a/g < s_0 + s_{10}$, there is no trajectory which goes from the region $s_3 > 0$ to the region $s_3 < 0$ (and conversely). Therefore, there is localization in the sense that the region 1, around the fixed point $\left\{\phi = 0, s_3 = +\sqrt{s_0^2 - a^2/4g^2}\right\}$, and region 2, around the fixed point $\left\{\phi = 0, s_3 = -\sqrt{s_0^2 - a^2/4g^2}\right\}$, are no more connected by any continuous trajectory. They are also disconnected from the region 0 around the central fixed point $\{\phi = \pi, s_3 = 0\}$.

Actually, when $gs_0 > a/2$, a separatrix appears to divide the Poincaré section into the three regions defined above. The equation of the separatrix is given by (25) with $A = g$ (i.e. $a\cos\phi = g\sqrt{s_0^2 - s_3^2}$). The upper and lower points of the separatrix are $\{\phi = \pi, s_3 = \pm\kappa\}$ as deduced from (19). The separatrix is clear in figure 1(*b*).

5. Tunneling of the Stokes parameters in the classical chaotic approach

Strictly speaking, tunneling cannot occur in the classical problem, as the evolution equations are deterministic, while quantum tunneling is allowed because of the fundamental fluctuations of the quantum variables. So, we will consider additional fluctuations of the physical parameters in the polarization problem. This is a drastic change for the classical system.

Indeed, without fluctuation of the parameters, the classical system is autonomous with two degrees of freedom, and, consequently, cannot exhibit any form of chaotic behavior according to the Poincaré–Benxidon theorem [18]. With fluctuations, it may become chaotic for some range of parameters, as we shall see below.

Several choices are possible (fluctuations of a, b, g, γ or any combination) and lead essentially to similar qualitative conclusions. We will focus below on a simple case, namely, when the Kerr parameter $a > 0$ is fixed (while $b = \gamma = 0$), and g, the nonlinear parameter, is modulated according to

$$g(z) = g_o + \tilde{g}\sin kz,$$

with $\tilde{g}$ a small coefficient and k the spatial frequency. Then, we have replaced the condition $g = g_o$, by $\langle g\rangle = g_o$, allowing for periodic modulation for this parameter. We will consider any positive value of k such that more complex (or even random) fluctuations of g could be obtained by the Fourier transform.

Note that the Hamiltonian is no more conservative as $\mathrm{d}H_2/\mathrm{d}z = \tilde{g}ks_3^2\cos kz$ (though, in average, $\langle \mathrm{d}H_2/\mathrm{d}z\rangle \equiv \lim_{Z\to\infty} 1/Z\int_0^Z \mathrm{d}H_2(z) = 0$).

5.1. *Example of chaotic tunneling*

Let us consider the system in the initial state $\{\phi, s_{30} = s_0\}$, with any value of ϕ. For the $\tilde{g}$ values small enough, the trajectory in the Poincaré section is a closed localized orbit, and the $s_3 < 0$-region is then unreachable.

When now $\tilde{g}$ is large enough, that is $\tilde{g}s_0 \geqslant 0.35$ for $g_os_0 = 3/2$ and $k = 1$, the system state can cross over the forbidden classical region (i.e., the forbidden region common to all the parameters g such that $g_o - \tilde{g} \leqslant g \leqslant g_o + \tilde{g}$), by chaotic behavior (see figure 2(*b*)) in order to reach the $s_3 < 0$-region. One can define a probability of transition as $P_{1\to 2}^{cl} = 1/\omega t_-$, where t_- is the lifetime in the $s_3 > 0$-region (that is the time needed for the system state to reach the $s_3 < 0$-region for the first time). Indeed, if the system stays indefinitely in the $s_3 > 0$-region, the probability $P_{1\to 2}^{cl} = 0$, while the probability equals 1 if it jumps to the $s_3 < 0$-region after one orbit.

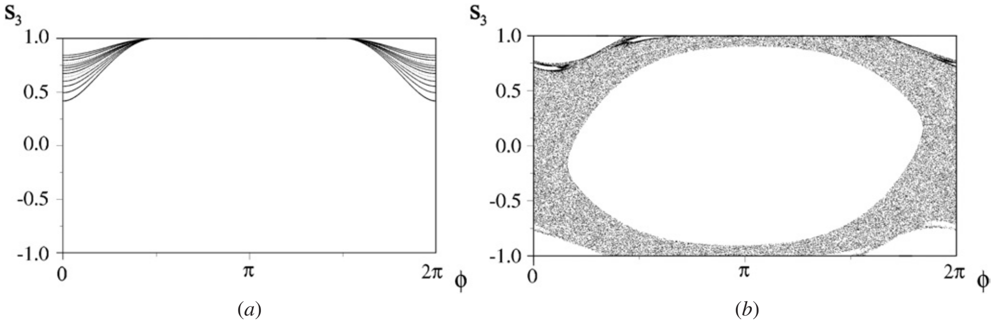

Figure 2. Poincaré sections for $s_0 = 1$ and (*a*) $(a, gs_0) = (1, g_o s_0)$, with $g_o s_0$ in between 1.1 and 1.9, by steps of 0.05 (continuous lines); (*b*) $(a, gs_0) = (1, 3/2 + 0.4 \sin z)$. Same parameters as in figure 1. Initial conditions: 1000 starting points $\{\phi, s_3 = 1\}$, random ϕ between 0 and π. Regular localization is clear in part (*a*) for the fixed value of $g_o s_0$. This will be the same when the value of $\tilde{g}$ is small enough for the system to be stable with respect to the small modulations of the nonlinear parameter g. In part (*b*), the value of $\tilde{g}$ is above the chaos threshold for $k = 1$, and the system-state crosses over the forbidden region to reach the negative values of s_3.

On the example of figure 2, that is $\tilde{g}s_0 = 0.4$, one obtains $P^{cl}_{1\to 2} \sim 6 \times 10^{-4}$. The limit case of interest is $\tilde{g}s_0 = 0.5$ as it corresponds to the situation where one of the possible orbits $g_o - \tilde{g} \leqslant g \leqslant g_o + \tilde{g}$ is marginally in the domain $s_3 > 0$ (namely, $gs_0 = 1$). For this limit case, the probability of classical transition appears to be $P^{cl}_{1\to 2} = 4 \times 10^{-2}$, of the same order of magnitude as the probability of quantum transition found in section 3.1. More generally (on the basis of other sets of parameters), one finds similar agreement between the values of $P^{qu}_{1\to 2}$ and $P^{cl}_{1\to 2}$ if one considers that the natural quantum fluctuations of the polarization correspond to the limit case of the classical fluctuations with localization.

6. Conclusion

We have investigated the optical gyration of the Stokes vector representing the polarization state of light propagating in a medium with constant Kerr effect and modulated nonlinearity. In particular, we focused on the possible tunneling process of the polarization state. When the average nonlinear parameter is large enough, even small fluctuations may induce chaotic behavior in the space evolution of the Stokes parameters. This results in dynamical chaotic tunneling across the polarization-state domains which are classically forbidden. Actually, cross-over appears between two localized orbits, because of the resonance occurring between the nonlinear and forced oscillations.

On the other hand, we have considered polarization tunneling by adopting a purely quantum mechanical idea. Here, tunneling is directly described as mutual change between two localized orbits on the Poincaré sphere, separated by a (classically) forbidden region. We obtain a concise analytic form of the tunneling probability by applying the functional integral approach in the Stokes parameters space.

Actually, nonlinearities are one of the most important issues of modern optics, as they deal with light controlled by light. The results presented here show how the stability of light polarization is very much concerned with the magnitude of the nonlinear parameters of the optical medium.

Acknowledgment

RB would like to thank the Ritsumeikan University for partial support concerning this work.

References

[1] Born M and Wolf E 1975 *Principle of Optics* (Oxford: Pergamon)
[2] Brosseau C 1998 *Fundamentals of Polarized Light* (New York: Academic)
[3] Kuratsuji H and Kakigi S 1998 *Phys. Rev. Lett.* **80** 1888
[4] Josephson B D 1962 *Phys. Lett.* A **1** 251
[5] Salmond G L, Holmes C A and Milburn G J 2002 *Phys. Rev.* A **65** 033623
[6] Botet R, Kuratsuji H and Seto R 2006 *Prog. Theor. Phys.* **116** 285
[7] van Hemmen J L, Hey H and Wreszinski W F 1997 *J. Phys. A: Math. Gen.* **30** 6371
[8] Seto R, Kuratsuji H and Botet R 2005 *Europhys. Lett.* **71** 751
[9] Kuratsuji H, Botet R and Seto R 2007 *Prog. Theor. Phys.* **117** 195
[10] Gasiorowitz S 1996 *Quantum Physics* 2nd edn (New York: Wiley)
[11] Akhmanov S A 1997 *Physical Optics* (Oxford: Clarendon) chapter 14
[12] Saleh B E A and Teich M C 1991 *Fundamentals of Photonics* (New York: Wiley)
[13] Landau L and Lifschitz E 1968 *Electrodynamics in Continuous Media* vol 8 (Oxford: Pergamon)
[14] Kuratsuji H and Suzuki T 1980 *J. Math. Phys.* **21** 472
[15] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[16] Lipkin H J, Meshov N and Glick A J 1965 *Nucl. Phys.* **62** 188
[17] Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions* (New York: Dover)
[18] Alligood K T, Sauer T D and Yorke J A 1997 *Chaos: An Introduction to Dynamical Systems* (New York: Springer)